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The self-dual string soliton in $AdS_4 \times S^7$ spacetime

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Abstract. We construct self-dual string soliton solutions in $AdS_4 \times S^7$ spacetime, starting from the covariant equations of motion of the M5-brane. We study the properties of the solutions and find that their actions are linearized, indicating the BPS nature of the solutions, and we find that they have the same electric and magnetic charges. The straight string soliton solution represents the configuration of the membranes ending on a M5-brane with a straight string intersection, and it behaves like the spiky solution in flat spacetime. The spherical string soliton solution, which could be related to the straight one by a conformal transformation, represents the membranes ending on a M5-brane with a spherical intersection.

Among the D-brane configurations, the spiky string is one of the most remarkable. It was first discussed in [1, 2]. From the DBI action of the D-brane, it was found that there could exist a BPS spiky solution, corresponding to the configuration of a fundamental string ending on the D-brane. In particular, for the D3-brane in IIB string theory, the spiky solution could also describe the D1-string or (p, q) string ending on the D3-brane, corresponding to the magnetic monopole or to dyons in $\mathcal{N} = 4$ super-Yang–Mills theory. On the other hand, from the dual D-string point of view, this spiky string configuration could be understood as the D-string funnel [3], whose non-commutative realization reflects the non-Abelian nature of the action of D-strings.

The spiky solution has a natural generalization in M-theory. The BPS self-dual string soliton solution constructed in [4] is its M-theory cousin. In this case, one has to solve the M5-brane's equations of motion, involving an interacting 2-form gauge field. The BPS soliton solution found in [4] represents a supersymmetric self-dual string on a M5-brane, with equal electric and magnetic charges. The configuration could be interpreted as the membranes ending on the M5-brane. The dual membrane description of the configuration was hindered by our ignorance of the non-Abelian membrane action. A few years ago, Basu and Harvey [5] proposed a generalized Nahm equation and constructed a funnel-like solution to realize this string soliton configuration. Briefly speaking, the membranes interact with each other and expand to become an orthogonal M5-brane. Though it is still an open issue how to derive the generalized Nahm equation from a non-Abelian membrane action, the self-dual string soliton solutions turn out to be quite valuable for us to understand the membrane and M5-brane dynamics in M-theory. For a nice review of M-theory branes and their dynamics, see [6].

Most of the study on D-branes was focused on branes in flat spacetime. In curved spacetime, it is generically difficult to find a BPS D-brane solution. Of particular interest are the D-branes in $AdS_5 \times S^5$ spacetime, which could be essential to understand the AdS/CFT correspondence. This issue has been addressed in [7]. Besides the BPS D-brane with fluxes discussed in [7], it was recently found that there exists another kind of BPS D-brane, which could be taken as the blow-up of the lower dimensional D-branes. This happens in the D-brane description of the BPS Wilson lines [8–10]. In this case, for the Wilson loop operator in the high rank representation, the corresponding fundamental string blows up to a higher dimensional D-brane, which could be understood as a dielectric brane [11, 12]. The same is true for the Wilson–'t Hooft operators [13]. In this case, the blow-up of the D(F)-strings not only involves the interactions between strings but also involves the interaction between strings and the background RR-fluxes.

In M-theory, the BPS branes in curved spacetime are even less clear. Of particular interest are the spacetime $AdS_7 \times S^4$ and $AdS_4 \times S^7$, both being the maximally supersymmetric configurations in 11D supergravity. They also arise as the near horizon geometry of the M5-brane and M2-brane supergravity soliton solution. More importantly they are the playgrounds of the AdS/CFT correspondence in M-theory. The study of the BPS brane configuration may shed light on the dynamics in M-theory. Unlike the case of D-branes in $AdS_5 \times S^5$, there is no systematic study of the BPS membrane or M5-brane configurations in these backgrounds, as far as we know. The trouble mainly comes from the complicated form of the equations of motion. In [14], it has been shown that there exist M5-brane self-dual string soliton solutions in $AdS_7 \times S^4$ spacetime, corresponding to the M5-brane description of the Wilson surface operators in high rank representation in the 6-dimensional $(2, 0)$ superconformal field theory. It

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realizes the picture that the membranes interacting among themselves and with the background 4-form flux blow up to M5-branes. The same configuration has also been discussed in [15]. However, it is not clear how to understand these configurations from the dual point of view of the membranes.

In this letter, we construct the self-dual string soliton solutions in an $AdS_4 \times S^7$ background. They are half-BPS but behave different from the solutions in [14]. The straight string soliton solution looks like the spiky solution in flat spacetime.

Let us start from the M5-brane covariant equations of motion in eleven dimensions, which was first proposed in [16] in the superembedding formalism [17, 18] and was then re-derived by requiring κ-symmetry of an open $M2$-brane ending on the M5-brane [19]. For other derivations from various actions, see [20–22]. The bosonic components of the equations include a scalar equation and a tensor equation. The scalar equation takes the form

$$G^{mn}\nabla_m \mathcal{E}_n^{\underline{c}} = \frac{Q}{\sqrt{-g}}\epsilon^{m_1\cdots m_6} \times \left(\frac{1}{6!}F^{\underline{a}}{}_{m_1\cdots m_6} + \frac{1}{(3!)^2}F^{\underline{a}}{}_{m_1m_2m_3}H_{m_4m_5m_6}\right)P_{\underline{a}}{}^{\underline{c}} \tag{1}$$

and the tensor equation is of the form

$$G^{mn}\nabla_m H_{npq} = Q^{-1}(4Y - 2(mY + Ym) + mYm)_{pq}\,. \tag{2}$$

Here our notation is as follows: the indices from the beginning (middle) of the alphabet refer to the frame (coordinate) indices, and the underlined indices refer to the target space ones.

Let us spend some time to explain the quantities in the above equations. There exists a self-dual 3-form field strength h_{mnp} on the M5-brane worldvolume. From it, we can define

$$k_m{}^n = h_{mpq}h^{npq}\,, \tag{3}$$
$$Q = 1 - \frac{2}{3}\mathrm{Tr}k^2\,, \tag{4}$$
$$m_p{}^q = \delta_p{}^q - 2k_p{}^q\,, \tag{5}$$
$$H_{mnp} = 4Q^{-1}(1+2k)_m{}^q h_{qnp}\,. \tag{6}$$

Note that h_{mnp} is self-dual with respect to the worldvolume metric but not H_{mnp}, which instead satisfies the Bianchi identity

$$\mathrm{d}H_3 = -\underline{F}_4 \tag{7}$$

where $\underline{F}_4$ is the pull-back of the target space 4-form flux. The induced metric is simply

$$g_{mn} = \mathcal{E}_m^{\underline{a}}\mathcal{E}_n^{\underline{b}}\eta_{\underline{ab}}\,, \tag{8}$$

where

$$\mathcal{E}_m^{\underline{a}} = \partial_m z^{\underline{m}} E_{\underline{m}}^{\underline{a}}\,. \tag{9}$$

Here $z^{\underline{m}}$ is the target spacetime coordinate, which is a function of worldvolume coordinate z through embedding, and $E_{\underline{m}}^{\underline{a}}$ is the component of a target space vielbein. From the induced metric, we can define another tensor

$$G^{mn} = \left(1 + \frac{2}{3}k^2\right)g^{mn} - 4k^{mn}\,, \tag{10}$$

which appears in (1). And we also have

$$P_{\underline{a}}{}^{\underline{c}} = \delta_{\underline{a}}^{\underline{c}} - \mathcal{E}_{\underline{a}}^m \mathcal{E}_m^{\underline{c}}. \tag{11}$$

Moreover, there is a 4-form field strength $F_{\underline{a}_1\cdots\underline{a}_4}$ and its Hodge dual 7-form field strength $F_{\underline{a}_1\cdots\underline{a}_7}$:

$$F_4 = \mathrm{d}C_3\,,$$
$$F_7 = \mathrm{d}C_6 + \frac{1}{2}C_3 \wedge F_4\,. \tag{12}$$

The frame indices on F_4 and F_7 in the scalar and the tensor equations have been converted to worldvolume indices with factors of $\mathcal{E}_m^{\underline{c}}$. From them, we can define

$$Y_{mn} = [4\star\underline{F} - 2(m\star\underline{F} + \star\underline{F}m) + m\star\underline{F}m]_{mn}\,, \tag{13}$$

where

$$\star\underline{F}^{mn} = \frac{1}{4!\sqrt{-g}}\epsilon^{mnpqrs}\underline{F}_{pqrs}\,. \tag{14}$$

One of the maximally supersymmetric configurations in 11-dimensional supergravity is the $AdS_4 \times S^7$ background, which could be obtained from the near horizon geometry of the supergravity solution of M2-branes. The metric and the bulk 4-form flux take the form

$$\mathrm{d}s^2 = \frac{R^2}{y^2}(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}r^2 + \mathrm{d}y^2) + 4R^2\mathrm{d}\Omega_7^2\,, \tag{15}$$

$$F_4 = -\frac{3R^3}{y^4}\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}r \wedge \mathrm{d}y\,, \tag{16}$$

where we have rescaled the radius of AdS_4 to be R and the radius of S^7 to be $2R$. Correspondingly the 4-form field strength gets rescaled by a factor 2.

Let the worldvolume coordinates of the M5-brane be ξ_i, $i = 0, \cdots, 5$ and the embedding be

$$\xi_0 = t\,, \quad \xi_1 = x\,, \quad \xi_2 = y\,, \quad r = f(y)\,, \tag{17}$$
$$\xi_3 = \alpha\,, \quad \xi_4 = \beta\,, \quad \xi_5 = \gamma\,, \tag{18}$$

where α, β, γ are the angular coordinates of a S^3 in S^7. In fact, it turns out that there is freedom to choose the embedding of S^3 in S^7, since the non-trivial equations of motions are actually in the AdS_4 part. For simplicity, we take the first three angular coordinates in S^7. The induced metric is then

$$\mathrm{d}s^2_{\mathrm{ind}} = \frac{R^2}{y^2}\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \left(1 + f'^2\right)\mathrm{d}y^2\right) + 4R^2\left(\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\beta^2 + \sin^2\alpha\sin^2\beta\,\mathrm{d}\gamma^2\right)\,, \tag{19}$$

where the prime denotes the partial derivative with respect to y.

The self-dual 3-form field strength could be

$$h = \frac{a}{2}\left((2R)^3 \sin^2\alpha \sin\beta \, \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma + \sqrt{1+f'^2}\left(\frac{R}{y}\right)^3 \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y\right), \quad (20)$$

where a could be a function of y. Then we can determine the other quantities, k^2, $k_m^{\ n}$ and G^{mn},

$$k^2 = k_{mn}k^{mn} = \frac{3}{2}a^4, \quad k_m^{\ n} = \begin{pmatrix} -\frac{a^2}{2}I_3 & 0 \\ 0 & \frac{a^2}{2}I_3 \end{pmatrix}, \quad (21)$$

$$G^{tt} = -G^{xx} = -\left(\frac{y}{R}\right)^2 c\left(1+a^2\right)^2,$$

$$G^{yy} = \left(\frac{y}{R}\right)^2 \frac{\left(1+a^2\right)^2}{1+f'^2}, \quad (22)$$

$$G^{\alpha\alpha} = \frac{\left(1-a^2\right)^2}{4R^2}, \quad G^{\beta\beta} = \frac{\left(1-a^2\right)^2}{4R^2\sin^2\alpha},$$

$$G^{\gamma\gamma} = \frac{\left(1-a^2\right)^2}{4R^2\sin^2\alpha\sin^2\beta}, \quad (23)$$

where I_3 is a rank 3 identity matrix. Also we have

$$H_3 = 2a\left(\frac{(2R)^3}{1-a^2}\sin^2\alpha\sin\beta\,\mathrm{d}\alpha\wedge\mathrm{d}\beta\wedge\mathrm{d}\gamma + \frac{\sqrt{1+f'^2}}{1+a^2}\left(\frac{R}{y}\right)^3 \mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}y\right). \quad (24)$$

Since the induced field strength of the 4-form bulk flux is vanishing, we have $\mathrm{d}H_3 = 0$, which requires a to be constant.

From the tensor equation of motion (2), we find that f has to be proportional to y,

$$f = \kappa y, \quad (25)$$

with κ being a constant. We list the relevant Levi-Civita connection of the induced metric (19) in the appendix. Now from the induced metric

$$\mathrm{d}s^2_{\mathrm{ind}} = \frac{R^2}{y^2}\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \left(1+\kappa^2\right)\mathrm{d}y^2\right) + 4R^2\left(\mathrm{d}\alpha^2 + \sin^2\alpha\ \mathrm{d}\beta^2 + \sin^2\alpha\sin^2\beta\,\mathrm{d}\gamma^2\right), \quad (26)$$

we find that M5-brane worldvolume is a $AdS_3 \times S^3$ spacetime, where AdS_3 with radius $R/\sqrt{1+\kappa^2}$ is embedded into AdS_4 and S^3 with radius $2R$ is completely in S^7.

In the scalar equations of motions, since the embedding of S^3 part is trivial, we always have $\nabla\mathcal{E}^{\underline{c}}_m = 0$ for the S^3 part. The right hand side of the equation of motions indeed vanishes, since $P^{\underline{c}}_{\underline{a}} = 0$ in this case.

The whole non-trivial part in the scalar equations of motion comes from the AdS_3 part. In this case, we have

$$\mathcal{E}^{\underline{0}}_t = \frac{R}{y}, \quad \mathcal{E}^{\underline{1}}_y = \frac{R}{y}, \quad \mathcal{E}^{\underline{2}}_x = \frac{R}{y}, \quad \mathcal{E}^{\underline{3}}_y = \frac{\kappa R}{y}, \quad (27)$$

where the vielbein of the AdS_4 part of the target spacetime is

$$\hat{\theta}^0 = \frac{R}{y}\mathrm{d}t, \quad \hat{\theta}^1 = \frac{R}{y}\mathrm{d}y, \quad \hat{\theta}^2 = \frac{R}{y}\mathrm{d}x, \quad \hat{\theta}^3 = \frac{R}{y}\mathrm{d}r. \quad (28)$$

The corresponding spin connections may be found in the appendix.

It turns out that the scalar equations of motion give only one constraint in this case,

$$\frac{(1+a^2)\kappa}{\sqrt{1+\kappa^2}} = 2a \quad (29)$$

or

$$a = \frac{\sqrt{1+\kappa^2}-1}{\kappa}. \quad (30)$$

Therefore, we have a self-dual string soliton solution in an $AdS_4 \times S^7$ background once the above relation is satisfied.

It would be interesting to study the properties of this string soliton solution. Firstly this solution is actually half-BPS. This may be seen by an appropriate coordinate transformation such that the above solution is actually the same configuration as discussed in [23], where the supersymmetry has been discussed in detail. The condition (30) is exactly the condition for keeping half of the supersymmetries. In fact, one may consider the more general background $AdS_4 \times X_7$, where X_7 is a weak G_2 manifold. The M5-brane worldvolume takes the form $AdS_3 \times L_3$, where L_3 is an associate submanifold in X_7 and AdS_3 is embedded in AdS_4 as above. Such a kind of M5-brane has been discussed in [23], where it has been shown to be BPS under suitable conditions. This kind of configuration gives also self-dual string soliton solutions. One piece of support for this claim is that the configurations are half-BPS. Though we are unaware of a direct proof that the BPS configurations must be the solutions of the equations of motion, the experience with the BPS Wilson lines and the Wilson surfaces indicates that this is true. One could also prove the claim by checking the equations of motion directly. Loosely speaking, since the L_3 embedding is somewhat trivial, the non-trivial part is from the AdS_3 part, which leads to the relation (30).

Let us come back to the solution in this paper. The action of this solution could be worked out. As is well known, compared to the Dp-brane action, which is just a Dirac–Born–Infeld (DBI)-type action, the M5-brane action is much subtler, since it describes a self-interacting chiral 2-form whose field strength is self-dual. There exist two kinds of action, which are equivalent in the sense that they lead to the same equations of motion. One of them is the so called PST (Pasti–Sorokin–Tonin) action [24–28]. It is manifestly supercovariant and kappa-invariant and of a DBI-like form. It contains an auxiliary scalar, from which the self-duality condition could be derived as an equation of motion. This proposal has some troubles in defining a proper partition function, since the topological class of the auxiliary scalar would break some symmetries of M-theory [29]. The resolution of this problem is to embed the chiral theory into a non-chiral one. In [21], a non-chiral M5-brane action for an unconstrained 2-form gauge potential has been constructed. In this action, one has to impose a non-linear self-duality condition to ensure kappa-symmetry. The equation of motion for the 2-form potential is equivalent to the Bianchi identity. The action is given by

$$S = S_{M5} - S_{\mathrm{WZ}} = T_5 \int \left(\frac{1}{2} \star \mathcal{K} - Z_6 \right), \quad (31)$$

where

$$\mathcal{K} = 2\sqrt{1 + \frac{1}{12}H^2 + \frac{1}{288}(H^2)^2 - \frac{1}{96}H_{abc}H^{bcd}H_{def}H^{efa}}\,, \quad (32)$$

$$Z_6 = \underline{C}_6 - \frac{1}{2}\underline{C}_3 \wedge H_3\,, \quad (33)$$

and T_5 is the tension of the M5-brane:

$$T_5 = \frac{1}{(2\pi)^5 l_{\mathrm{P}}^6}\,. \quad (34)$$

Here Z_6 is the Wess–Zumino term, in which $\underline{C}_6$ and $\underline{C}_3$ are the pull-backs of the target space gauge potential. For the self-dual soliton solution above, classically two kinds of action give the same action:

$$S = \int \mathrm{d}(\mathrm{Vol})(1+\kappa^2)\,, \quad (35)$$

where $\mathrm{d}(\mathrm{Vol})$ is the volume element of M5-brane without flux. In other words, if we turn off the self-dual field strength, $a = 0$, then we have $\kappa = 0$, which implies that we have a trivial embedding of the M5-brane into the background. In this case, the M5-brane is $AdS_3 \times S^3$ with radius R in AdS_3 and radius $2R$ in S^3. Therefore,

$$\mathrm{d}(\mathrm{Vol}) = \left(\frac{R}{y}\right)^3 (2R)^3 \sin^2\alpha \sin\beta\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\,. \quad (36)$$

The two terms in the integrand in (35) have a natural interpretation: the first term gives the action of the M5-brane without 3-form field strength, the second term gives the action of the membrane ending on the M5-brane. The absence of the square root in the action indicates that the solution is BPS. This is reminiscent of the BPS spiky solution studied in [1, 2], which has an action similar to the one above. This is very similar to the case in flat spacetime [4].

Let us check the charges carried by the string soliton. Since our solution could be taken as M2-branes ending on a M5-brane, with M2-branes' worldvolume extending along t, x, r, the charges could be calculated by

$$Q_E = \frac{1}{\mathrm{Vol}(S^3)} \int_{S^3} \star H\,, \quad (37)$$

$$Q_M = \frac{1}{\mathrm{Vol}(S^3)} \int_{S^3} H\,, \quad (38)$$

where S^3 is the transverse S^3 in S^7 and $\star$ here means the Hodge dual with respect to the metric of the M5-brane worldvolume without the string soliton. The charge of our solution is simply

$$Q_M = \left(\frac{2R}{l_{\mathrm{P}}}\right)^3 \frac{2a}{1-a^2} = \left(\frac{2R}{l_{\mathrm{P}}}\right)^3 \kappa\,, \quad (39)$$

$$Q_E = \left(\frac{2R}{l_{\mathrm{P}}}\right)^3 \frac{2a\sqrt{1+\kappa^2}}{1+a^2} = \left(\frac{2R}{l_{\mathrm{P}}}\right)^3 \kappa\,. \quad (40)$$

So our solution has the same electric and magnetic charges. This is reminiscent of the self-dual soliton solution in flat spacetime [4]. In terms of the charge, one can see that the action of the membrane of unit charge is the membrane tension times its volume.

It is remarkable that although our solution looks at first sight similar to the ones in flat spacetime and the ones corresponding to Wilson surface operators in $AdS_7 \times S^4$ [14], they are really different. Compared to the flat spacetime case, our solution depends on the background 4-form field strength, which makes the spacetime curved. In the Wilson surface case, the M5-brane could be taken as the blow-up of the membrane interaction and its worldvolume could collapse once the membrane charge is turned off, while in the solution we studied in this paper, the M5-brane could still make sense even if the membrane charge were turned off. This difference is reminiscent of the difference between the D-brane description of the Wilson–'t Hooft operators [8, 13] and the BPS brane with fundamental fluxes in $AdS_5 \times S^5$ spacetime [7].

Let us consider the perturbation around the string soliton solution. For simplicity, we just focus on the one along the α direction. The perturbation satisfies the following equation:

$$G^{mn}\nabla_m \partial_n \delta = 0\,, \quad (41)$$

where δ is the perturbation and G^{mn} is the tensor defined in (10). Since G^{mn} is diagonal and one can separate the S^3 part and simplify the above relation to become

$$\left(-\partial_t^2 + \partial_x^2 + \frac{1}{1+\kappa^2}\left(\partial_y^2 - \frac{1}{y}\partial_y\right) + \frac{A}{y^2}\right)\delta = 0\,, \quad (42)$$

where A is the contribution from the S^3 part, involving the angular momenta quantum numbers. For simplicity, we may take the S-wave case so that $A = 0$. The above equation could be simplified more by setting $\delta = \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}k_x x}\delta_y$. Then we have

$$\left(\partial_y^2 - \frac{1}{y}\partial_y + \left(1+\kappa^2\right) B\right)\delta_y = 0\,, \quad (43)$$

where $B = \omega^2 - k_x^2$. Without losing generality, setting $k_x = 0$ and defining $\rho = y\omega$, we find

$$\left(\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \left(1+\kappa^2\right)\right)\delta_\rho = 0\,. \quad (44)$$

The solution to this equation is

$$\delta_\rho = \rho\left(A\, J_1\left(\sqrt{1+\kappa^2}\rho\right) + BN_1\left(\sqrt{1+\kappa^2}\rho\right)\right)\,, \quad (45)$$

where J_1 and N_1 are the Bessel function and Neumann function, respectively. The asymptotic behavior of the solution could be discussed straightforwardly. It is remarkable that the perturbation in our case could be solved exactly, unlike the spiky string case [1]. Near $\rho \to 0$ or $y \to 0$, the Neumann function blows up, so we just choose the Bessel function above.

The solution we discussed above could be understood in terms of membranes ending on the M5-brane with a straight string-like intersection. The other worldvolume direction of the membranes is an infinitely straight line extending along r. Actually one can show that through a conformal transformation, one can obtain another M5-brane configuration with spherical intersection. This is more easily seen in Euclidean signature. Since the embedding of S^3 in S^7 is somewhat trivial, our discussion will just focus on the AdS_4 part. The metric of the Euclideanized AdS_4 could be written as

$$\mathrm{d}s^2 = \frac{R^2}{y^2}\left(\mathrm{d}y^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\beta^2\right)\right). \quad (46)$$

Introducing the coordinates η, ρ by the following relations:

$$r = \frac{R\cos\eta}{\cosh\rho - \sinh\rho}, \qquad y = \frac{R\sin\eta}{\cosh\rho - \sinh\rho}, \quad (47)$$

we can rewrite the metric as

$$\mathrm{d}s^2 = \frac{R^2}{\sin^2\eta}\left(\mathrm{d}\eta^2 + \cos^2\eta\left(\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\beta^2\right) + \mathrm{d}\rho^2\right). \quad (48)$$

Inspired by the study of the Wilson surface operators in [14], we make the following ansatz:

$$\sin\eta = \kappa^{-1}\sinh\rho, \quad (49)$$

so the induced metric of the M5-brane worldvolume is

$$\mathrm{d}s^2 = \frac{R^2}{\sin^2\eta}\left(\frac{1+\kappa^2}{1+\kappa^2\sin^2\eta}\mathrm{d}\eta^2 + \cos^2\eta\left(\mathrm{d}\alpha^2 + \sin^2 a\ \mathrm{d}\beta^2\right)\right). \quad (50)$$

It is straightforward to check that the above configuration is indeed a solution of the equations of motion provided that κ satisfy the relation (30). The charges of the solution are the same as in the case of the straight one. However, the action of the solution looks more involved. In fact, the 3-form gauge potential could be taken as

$$C_3 = -R^3\frac{\cos^2\eta}{\sin^3\eta}\left(-\sin\eta + \frac{\kappa\cos^2\eta}{\sqrt{1+\kappa^2\sin^2\eta}}\right) \times \sin\alpha\,\mathrm{d}\eta\wedge\mathrm{d}\alpha\wedge\mathrm{d}\beta. \quad (51)$$

The non-chiral part of the action is the same as the straight case, but the Chern–Simons coupling depends on C_3. The action consists of two parts: one part giving the action of a M5-brane without flux, and the other part showing the contribution of the string soliton flux. The absence of the square root in the action indicates the BPS nature of the configuration. The difference of the flux energy with the straight one may come from the conformal anomaly discussed in [30].

In this letter, we started from the covariant equation of motion of a M5-brane and constructed the BPS soliton solutions. For the straight soliton solution we constructed, it is the same configuration as discovered in [23], where it has been proved to be supersymmetric. The same configuration has been discussed in [15] from the PST action. However, in our work we were more concerned with the physical properties of the configuration. We showed that its action consists of two parts: one part is the M5-brane action without flux, the other one is the contribution from the flux. The action is linearized, indicating its BPS nature. We also showed that the solution has the same electric and magnetic charge. It is quite different from the M5-brane self-dual string soliton solutions discussed in [14]. The essential difference lies in the fact that the M5-brane configurations in [14] are the blow-up of the membrane and could collapse once the membrane charge is turned off, while in the configuration discussed here the M5-brane is well embedded in the curved background even without membrane charge. Both the electric and magnetic charges of the membranes in this configuration are well defined in the sense of [4], while for the configurations in [14], the definition of the electric charge is subtle. The solution here looks similar to the spiky solution discussed in [1]. But in our case, the construction of the solution involves the background flux. Furthermore, we showed that through a conformal transformation, we have a new string soliton solution, representing the membranes ending on a M5-brane with a spherical intersection.

One interesting issue is to find a dual description from the non-Abelian membrane interaction [5]. However, this seems to be quite difficult. One obstacle is that we do not know how to describe the membrane dynamics from the point of view of a non-Abelian membrane action even in flat spacetime [31], not to mention in curved spacetime. Furthermore, in our background we have to consider the coupling of membranes with the background 4-form flux. In the D-brane case, the non-Abelian Chern–Simons term governing the coupling between D-brane and the background RR gauge potential leads to the dielectric branes [11]. In M-theory, we do not know how to generalize the Chern–Simons coupling to the non-Abelian case. We wish to come back to this issue in the future.

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Appendix : Various connections

In this appendix, we list various connections appearing in our calculation. For the induced metric (19), its Christoffel symbol has the non-vanishing components

$$\begin{aligned}
\Gamma^t_{yt} &= \Gamma^x_{yx} = -\frac{1}{y}\,, \\
\Gamma^y_{tt} &= -\Gamma^y_{xx} = -\frac{1}{y}\frac{1}{1+f'^2}\,, \\
\Gamma^y_{yy} &= -\frac{1}{y} + \frac{y'y''}{1+y'^2}\,, \\
\Gamma^\alpha_{\beta\beta} &= -\sin\alpha\cos\alpha\,, \\
\Gamma^\alpha_{\gamma\gamma} &= -\sin^2\beta\sin\alpha\cos\alpha\,, \\
\Gamma^\beta_{\alpha\beta} &= \frac{\cos\alpha}{\sin\alpha}\,, \\
\Gamma^\beta_{\gamma\gamma} &= -\sin\beta\cos\beta\,, \\
\Gamma^\gamma_{\alpha\gamma} &= \frac{\cos\alpha}{\sin\alpha}\,, \\
\Gamma^\gamma_{\beta\gamma} &= \frac{\cos\beta}{\sin\beta}\,.
\end{aligned} \tag{A.1}$$

When $f = \kappa r$, some of the components above are vanishing.

For the AdS_4 spacetime, its non-vanishing independent components of the spin connection are

$$\omega^{\underline{1}}_{\underline{00}} = -\frac{1}{R}\,, \quad \omega^{\underline{1}}_{\underline{22}} = \omega^{\underline{1}}_{\underline{33}} = \frac{1}{R}\,. \tag{A.2}$$

References

1. C.G. Callan, J.M. Maldacena, Nucl. Phys. B **513**, 109 (1998)
2. G.W. Gibbons, Nucl. Phys. B **514**, 603 (1998)
3. N.R. Constable, R.C. Myers, O. Tafjord, Phys. Rev. D **61**, 106 009 (2000) [hep-th/9911136]
4. P.S. Howe, E. Sezgin, P.C. West, Phys. Lett. B **399**, 49 (1997)
5. A. Basu, J. Harvey, Nucl. Phys. B **713**, 136 (2005) [hep-th/0412310]
6. D.S. Berman, arXiv:0710.1707
7. K. Skenderis, M. Taylor, JHEP **0206**, 025 (2002)
8. N. Drukker, B. Fiol, JHEP **02**, 010 (2005)
9. S. Yamaguchi, JHEP **0605**, 037 (2006)
10. J. Gomis, F. Passerini, JHEP **0608**, 074 (2006) [hep-th/0604007]
11. R.C. Myers, JHEP **9912**, 022 (1999) [hep-th/9910053]
12. D. Rodriguez-Gomez, Nucl. Phys. B **752**, 316 (2006) [hep-th/0604031]
13. B. Chen, W. He, Phys. Rev. D **74**, 126 008 (2006) [hep-th/0607024]
14. B. Chen, W. He, J.B. Wu, L. Zhang, JHEP **0708** (2007) [arXiv:0707.3978]
15. O. Lunin, arXiv:0704.3442
16. P.S. Howe, E. Sezgin, P.C. West, Phys. Lett. B **399**, 49 (1997) [hep-th/9702008]
17. P.S. Howe, E. Sezgin, Phys. Lett. B **390**, 133 (1997) [hep-th/9607227]
18. P.S. Howe, E. Sezgin, Phys. Lett. B **394**, 62 (1997) [hep-th/9611008]
19. C.S. Chu, E. Sezgin, JHEP **12**, 001 (1997) [hep-th/9710223]
20. E. Sezgin, P. Sundell, hep-th/9902217
21. M. Cederwall, B.E.W. Nilsson, P. Sundell, hep-th/9712059
22. I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin, M. Tonin, Phys. Lett. B **408**, 135 (1997) [hep-th/9703127]
23. A. Yamaguchi, hep-th/0305007
24. P. Pasti, D. Sorokin, M. Tonin, Phys. Lett. B **398**, 41 (1997) [hep-th/9701037]
25. I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin, M. Tonin, Phys. Rev. Lett. **78**, 4332 (1997) [hep-th/9701149]
26. M. Aganagic, J. Park, C. Popescu, J.H. Schwarz, Nucl. Phys. B **496**, 191 (1997) [hep-th/9701166]
27. J. Schwarz, Phys. Lett. B **395**, 191 (1997) [hep-th/9701008]
28. M. Perry, J. Schwarz, Nucl. Phys. B **489**, 47 (1997) [hep-th/9611065]
29. E. Witten, J. Geom. Phys. **22**, 103 (1997) [hep-th/9610234]
30. R. Graham, E. Witten, Nucl. Phys. B **546**, 52 (1999) [hep-th/9901021]
31. J. Bagger, N. Lambert, Phys. Rev. D **75**, 045 020 (2007) [hep-th/0611108]